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FREE VIBRATIONS OF SPINNING COMPOSITE CYLINDRICAL SHELLS

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Abstract—The free vibrations of rotating laminated filament-wound cylindrical shells have been investigated. The exact solution procedure was formulated for general field equations and general boundary conditions, arbitrary combinations of lamina materials, and fiber orientation. A parametric investigation of the free vibrations' spectra has been carried out. The main characteristics of spinning composite shells are presented and discussed as functions of the filament-winding angles, various layups and the rotational velocity.

INTRODUCTION

The dynamic behavior of spinning shells has been investigated for over a century. Since the appearance of Bryan's (1890) paper first discovering the traveling-modes phenomenon, numerous studies have been published. In particular, the following studies are noted: Carrier (1945), Di Taranto and Lessen (1964), Fox and Hardie (1985), Huang and Soedel (1988a,b), Padovan (1973) and Soedel (1976, 1981).

There are various engineering applications of spinning cylindrical shells—namely where the angular velocity vector coincides with the shell axis—such as high-speed centrifugal separators and gas turbines for high-power aircraft engines. Other applications are associated with spinning satellite structures and similar spacecrafts.

Until recently most of the research efforts had been devoted to the dynamic behavior of rotating *isotropic* shells. A review of analytical methods used to determine the modal characteristics of non-rotating cylindrical shells may be found in Forsberg's report (1966). The free and forced vibrations of *spinning* isotropic shells were treated by several investigators, in particular it is worth mentioning the work of Huang and Soedel (1988b). Their analysis of the free and forced vibrations of simply-supported rotating *isotropic* cylindrical shells resulted with the conclusion that the effect of rotation is mainly in bifurcating the natural frequencies into two branches of forward and backward waves.

The vibration analysis of *anisotropic composite* shells is of importance in view of the current interest in designing with composite materials. Not too many investigations dealt with this new field of application. The work of Greenberg and Stavsky (1981) is noted, presenting the vibration analysis of non-rotating laminated composite cylindrical shells.

Padovan (1975) was seemingly the first to consider the effects of material anisotropy on spinning cylindrical shells. In this investigation perturbational finite-element approximations were used to study the frequency and buckling eigenvalues of prestressed shells.

In what follows, a general formulation of the dynamic behavior of spinning composite cylindrical shells—within the framework of Love's shell theory—is given for arbitrary boundary conditions. The linear governing field equations are formulated for small vibratory motion around the rotated equilibrium state. The equations of motion in terms of displacements are expressed by "structural" and "dynamic" operators. A closed-form solution is given for the presented linear system of homogeneous differential equations. Finally, a parametric study of the free vibrations spectra of rotating shells is presented and discussed.

THEORY

Notation

The cylindrical shell under discussion is shown in Fig. 1a. The rotation is expressed by the velocity vector $\vec{\Omega}$ given by $\Omega\hat{x}$ where Ω is a scalar that may represent both positive and negative values.

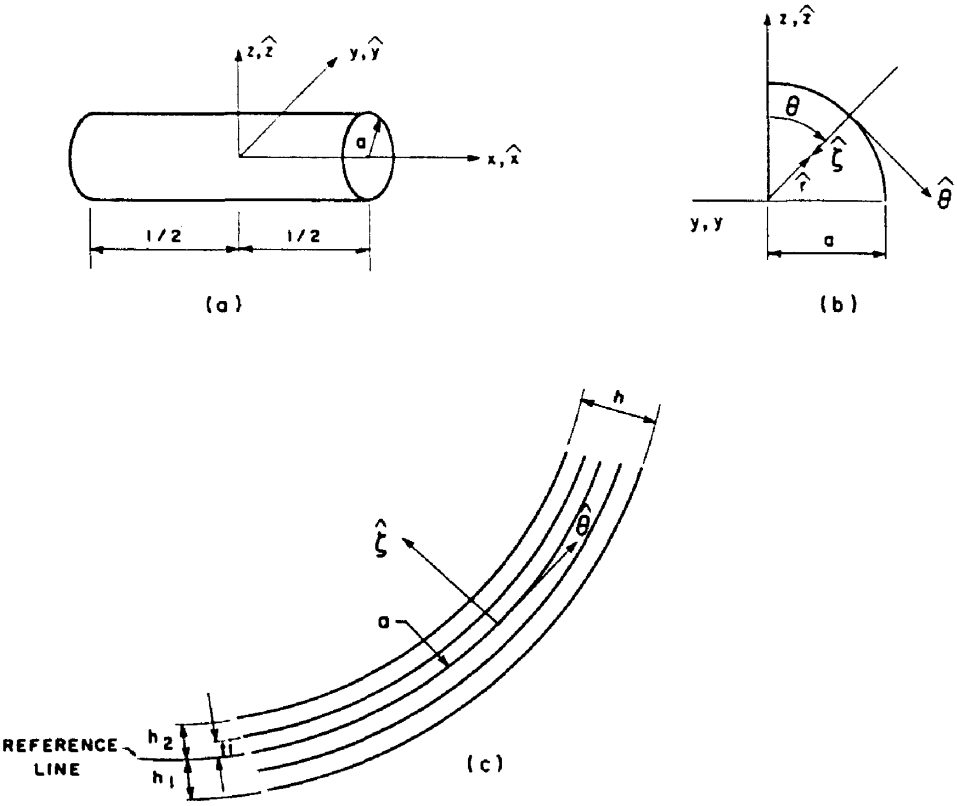


Fig. 1. The cylindrical shell geometry.

The non-dimensional (with respect to the shell radius a) axial coordinate is denoted by $\xi\left(=\frac{x}{a}\right)$, the circumferential direction is denoted by $\hat{\theta}$ and the direction normal to the surface by $\hat{\zeta}$ (see Fig. 1b). The shell is assumed to be made of layers of filament-wound orthotropic materials, the thicknesses of which are denoted by t_i (see Fig. 1c).

Equations of motion

The three-dimensional equations of motion for an infinitesimal element in polar coordinates *in the rotating frame* may be written, for the case of time-independent density, as follows [see e.g. Stavsky and Loewy (1971)]:

$$\tau_{rr,r} + \frac{1}{r}\tau_{\theta r,\theta} + \tau_{zr,z} + (\tau_{rr} - \tau_{\theta\theta})/r = \rho a_r \quad (1a)$$

$$\tau_{r\theta,r} + \frac{1}{r}\tau_{\theta\theta,\theta} + \tau_{z\theta,z} + \frac{2}{r}\tau_{r\theta} = \rho a_\theta \quad (1b)$$

$$\tau_{rz,r} + \frac{1}{r}\tau_{\theta z,\theta} + \tau_{zz,z} + \frac{1}{r}\tau_{rz} = \rho a_z \quad (1c)$$

where r is a radial coordinate and a_r, a_θ and a_z are the components of the acceleration vector in the rotating frame given in the $\hat{r}$, $\hat{\theta}$, $\hat{z}$ directions, respectively. By dynamic considerations it is possible to show that:

$$a_r = \ddot{u}_r - \Omega^2(\underline{\underline{a}} + u_r) - \underline{2\Omega\dot{u}_\theta} \quad (2a)$$

$$a_\theta = \ddot{u}_\theta - \Omega^2 u_\theta + \underline{2\Omega\dot{u}_r} \quad (2b)$$

$$a_z = \ddot{u}_z \quad (2c)$$

where u_r, u_θ and u_z are the elastic displacements in the $\hat{r}$, $\hat{\theta}$, $\hat{z}$ directions, respectively, and $(\dot{\ })$ denotes differentiation with respect to time. Note that the terms underlined with a single line are those stemming from Coriolis acceleration and the one with a double line is an inertia term which is independent of the elastic deformation. Assuming linear displacement fields across the shell thickness [see Stavsky and Loewy (1971)], integrating eqns (1a–c) and substituting the more convenient notation $u = u_z$, $v = u_\theta$, $w = -u_r$, $dr = -d\zeta$, $dx = dz = a\, d\xi$, yields three force equations (see Appendix A for the definition of stress resultants and inertia terms):

$$N_{\theta\xi,\theta} + N_{\xi,\xi} - \bar{N}_\theta \frac{1}{a}(v_{,\xi\theta} - w_{,\xi}) = aR_0 a_\xi \tag{3a}$$

$$\bar{N}_{\theta,\theta} + N_{\xi\theta,\theta} - Q_\theta = aR_0 a_\theta \tag{3b}$$

$$Q_{\xi,\xi} + Q_{\theta,\theta} + \bar{N}_\theta\left(1 + \frac{1}{a}v_{,\theta} + \frac{1}{a}w_{,\theta\theta}\right) = aR_0 a_\zeta \tag{3c}$$

where it was assumed that due to the rotation, the circumferential force, $\bar{N}_\theta$, is very large in comparison with the other stress resultants and therefore its product with the displacements has been retained. $\bar{N}_\theta$ is decomposed to the sum of the initial value N_θ^i and the vibratory force N_θ. Assuming that the boundary conditions are enforced only after spinning has been attained, it can be shown that:

$$N_\theta^i = R_0 a^2 \Omega^2. \tag{4}$$

In addition, $\bar{N}_\theta$ should be replaced by $\bar{N}_\theta\left(1 + \frac{1}{a}u_{,\xi}\right)$ in the second and third equations of motion to account for the stretching of the middle surface. Retaining linear terms eqns (3a–c) take the form:

$$N_{\theta\xi,\theta} + N_{\xi,\xi} = aR_0 a_\xi + R_0 a^2 \Omega^2 (v_{,\xi\theta} - w_{,\xi}) \tag{5a}$$

$$N_{\theta,\theta} + N_{\xi\theta,\theta} - Q_\theta = aR_0 a_\theta - R_0^2 a^2 \Omega u_{,\xi\theta} \tag{5b}$$

$$Q_{\xi,\xi} + Q_{\theta,\theta} + N_\theta = aR_0 a_\zeta - R_0 a^2 \Omega^2 \left(1 + \frac{1}{a}v_{,\theta} + \frac{1}{a}w_{,\theta\theta} + \frac{1}{a}u_{,\xi}\right). \tag{5c}$$

In addition, multiplying eqns (2b–c) by $\zeta\, d\zeta$ and integrating, yields a pair of moment equations from which Q_θ and Q_ξ are obtained and substituted in eqns (4b,c).

Stress–strain relations are based on Hooke's law for an aeolotropic shell. In terms of the elastic stiffness moduli it is possible to express the stress–strain relations as:

$$\begin{bmatrix} \tau_\xi \\ \tau_\theta \\ \tau_{\xi\theta} \end{bmatrix} = \begin{bmatrix} E_{\xi\xi} & E_{\xi\theta} & E_{\xi s} \\ & E_{\theta\theta} & E_{\theta s} \\ \text{symm.} & & E_{ss} \end{bmatrix} \begin{bmatrix} \varepsilon_\xi \\ \varepsilon_\theta \\ \varepsilon_{\xi\theta} \end{bmatrix}. \tag{6}$$

Assuming linear strain variation across the shell thickness enables the strains to be expressed as functions of the strains at the reference surface $\varepsilon_\xi^0, \varepsilon_\theta^0, \varepsilon_{\xi\theta}^0$ and the curvature changes κ_ξ, κ_θ, $\kappa_{\xi\theta}$ by:

$$(\varepsilon_{\xi}, \varepsilon_{\theta}, \varepsilon_{\xi\theta}) = (\varepsilon^0_{\xi}, \varepsilon^0_{\theta}, \varepsilon^0_{\xi\theta}) + \zeta(\kappa_{\xi}, \kappa_{\theta}, \kappa_{\xi\theta}). \tag{7}$$

Consequently, based on the definitions in Appendix A, the shell stress–strain relations become:

$$\begin{bmatrix} N_{\xi} \\ N_{\theta} \\ N_{\xi\theta} \\ M_{\xi} \\ M_{\theta} \\ M_{\xi\theta} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{\xi\xi} & A_{\xi\theta} & A_{\xi s} & B_{\xi\xi} & B_{\xi\theta} & B_{\xi s} \\ A_{\theta\xi} & A_{\theta\theta} & A_{\theta s} & B_{\theta\xi} & B_{\theta\theta} & B_{\theta s} \\ A_{s\xi} & A_{s\theta} & A_{ss} & B_{s\xi} & B_{s\theta} & B_{ss} \\ \hline B_{\xi\xi} & B_{\xi\theta} & B_{\xi s} & D_{\xi\xi} & D_{\xi\theta} & D_{\xi s} \\ B_{\theta\xi} & B_{\theta\theta} & B_{\theta s} & D_{\theta\xi} & D_{\theta\theta} & D_{\theta s} \\ B_{s\xi} & B_{s\theta} & B_{ss} & D_{s\xi} & D_{s\theta} & D_{ss} \end{array}\right] \begin{bmatrix} \varepsilon^0_{\xi} \\ \varepsilon^0_{\theta} \\ \varepsilon^0_{\xi\theta} \\ \kappa_{\xi} \\ \kappa_{\theta} \\ \kappa_{\xi\theta} \end{bmatrix}. \tag{8}$$

In addition, the curvature terms may be expressed in terms of the displacements of the reference surface u^0, v^0, w^0 by:

$$\kappa_{\xi} = -\frac{1}{a^2} w^0_{,\xi\xi} \tag{9a}$$

$$\kappa_{\theta} = -\frac{1}{a^2}(v^0_{,\theta} + w^0_{,\theta\theta}) \tag{9b}$$

$$\kappa_{\xi\theta} = -\frac{1}{a^2}(v^0_{,\xi} + w^0_{,\xi\theta}). \tag{9c}$$

Substitution of eqns (9a–c) into eqn (7) and eqns (7) and (9a–c) into eqn (8) yields the stress–resultant-displacement relations.

Displacement equations of motion

The vibratory motion of the entire system is assumed to be harmonic with the same period, so that the time and spatial variations may be separated as:

$$(u, v, w) = (U_{1(\xi)}, U_{2(\xi)}, U_{3(\xi)})\, e^{i(\omega t + n\theta)} \tag{10}$$

where ω is the vibratory frequency and n is the circumferential mode. Note that in the following discussion, n is assumed to be known.

Substituting the above-mentioned stress–resultant-displacement relations in eqns (4a–c) enables, with the aid of eqn (10), the following linear system of equations to be formulated:

$$[L]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} q_{\xi} \\ q_{\theta} \\ q_{\zeta} \end{Bmatrix} \tag{11}$$

where $[L]$ contains "structural operators", $[L_S]$, which depend on the elastic stiffness moduli and geometrical properties and "rotational" operators, $[L_R]$, which depend on the angular velocity components, the natural frequency, geometrical properties and the density distribution. q_{ξ}, q_{θ} and q_{ζ} are given external forcing terms. The operators $[L_S]$ and $[L_R]$ are shown in Appendix B. It should be noted that the format of eqn (11) is generic and may be kept in case of other eighth-order shell theories.

In order to investigate the case of free vibrations, one should assume $q_{\xi} = q_{\theta} = q_{\zeta} = 0$ and eqns (11) become a linear homogeneous system of equations.

Method of solution

Since the functional operators are linear with constant coefficients, it is possible to express the displacements as a displacement function $\phi(\xi)$ [see Greenberg and Stavsky (1981)]:

$$[U_1, U_2, U_3] = [\mathscr{D}_{31}, \mathscr{D}_{32}, \mathscr{D}_{33}]\phi(\xi) \tag{12}$$

where $\mathscr{D}_{ij}$ are the minors of det (L_{ij}) and the displacement function ϕ is the solution of the homogeneous equation

$$\det(L_{ij})\phi = 0. \tag{13}$$

Generally, the operator det (L_{ij}) is eighth-order so the solution of eqn (13) is of the form:

$$\phi = \sum_{k=1}^{8} r_k \mathrm{e}^{\mu_k \xi} \tag{14}$$

where μ_k are the roots of the eighth-degree polynomial auxiliary equation and r_k are constants to be determined by the boundary conditions which are also written in terms of r_k and μ_k.

Since each type of boundary condition is associated with a combination of resultant forces, moments, displacements and their derivatives, it is always possible to express the forces and moments in terms of the displacements by:

$$\{N\} = [N_u]\{u\} \tag{15a}$$

where

$$\{N\}^T = \langle N_\xi, N_\theta, N_{\xi\theta}, M_\xi, M_\theta, M_{\xi\theta}, Q_\xi, Q_\theta \rangle \tag{15b}$$

$$\{u\}^T = \langle U_1, U_{1,\xi}, U_{1,\xi\xi}, U_2, U_{2,\xi}, U_{2,\xi\xi}, U_3, U_{3,\xi}, U_{3,\xi\xi} \rangle. \tag{15c}$$

Consequently, for each type of boundary condition it is possible to construct a matrix, $[R]$, which consists of the appropriate rows of $[N_u]$. For example, for the case of a shell which is simply-supported, of type SS3, at its two ends [i.e. $N_\xi = M_\xi = v = w = 0$ for $\xi = (0, l/a)$] one may write:

$$\begin{Bmatrix} N_\xi \\ M_\xi \\ v \\ w \end{Bmatrix} = [R]\{u\}. \tag{16}$$

The displacements vector $\{u\}$ is related to r_k and μ_k by the equation:

$$\{u\} = [P]\{r\} \tag{17}$$

where the elements of the matrix $[P]$ are given by:

$$P_{ij} = \left(\sum_{k=1}^{9} d_{31}(k)\mu_j^{k+i-2}\right)\mathrm{e}^{\mu_j \xi} \quad (i = 1, 3, j = 1, 8) \tag{18a}$$

$$P_{ij} = \left(\sum_{k=1}^{9} d_{32}(k)\mu_j^{k+i-5}\right)\mathrm{e}^{\mu_j \xi} \quad (i = 4, 6, j = 1, 8) \tag{18b}$$

$$P_{ij} = \left(\sum_{k=1}^{9} d_{33}(k)\mu_j^{k+i-8}\right) e^{\mu_j \xi} \quad (i = 7, 9, j = 1, 8). \tag{18c}$$

The $d_{ij}(k)$ elements are the coefficients of the $\mathscr{D}_{ij}$ operator, namely:

$$\mathscr{D}_{ij} = \sum_k d_{ij}(k) \frac{d^{k-1}(\ )}{d\xi^{k-1}} \tag{19}$$

and the vector $\{r\}$ is defined by:

$$\{r\}^T = \langle r_1, r_2 \ldots r_8 \rangle. \tag{20}$$

Thus, combining eqns (16), (17) enables us to write (for the case of a simply-supported shell):

$$\begin{Bmatrix} N_{\xi(0)} \\ M_{\xi(0)} \\ v_{(0)} \\ w_{(0)} \\ N_{\xi(1)} \\ M_{\xi(1)} \\ v_{(1)} \\ w_{(1)} \end{Bmatrix} = \begin{bmatrix} [R]\cdot[P_{(\xi=0)}] \\ \hline [R]\cdot[P_{(\xi=l/a)}] \end{bmatrix} \{r\} = [T]\{r\}. \tag{21}$$

The solution procedure is therefore initiated by assuming a value for the vibration frequency ω. Then, eqn (13) is solved and the roots μ_k are obtained. At this stage, the determinant of the matrix $[T]$ is calculated. This procedure is repeated until the value of ω that causes the vanishing of the determinant of $[T]$ is obtained. For that frequency, the eigenvalues of $\{r\}$ are calculated and ϕ, U_1, U_2 and U_3 are determined by eqns (14), (12).

Figure 2 presents a typical variation of the determinant of the matrix $[T]$ with frequency (values are normalized). As shown, the imaginary part dominates the problem in this case and its vanishing at $\omega = 820$ rad s^{-1} and $\omega = -2000$ rad s^{-1} represent the first two natural frequencies.

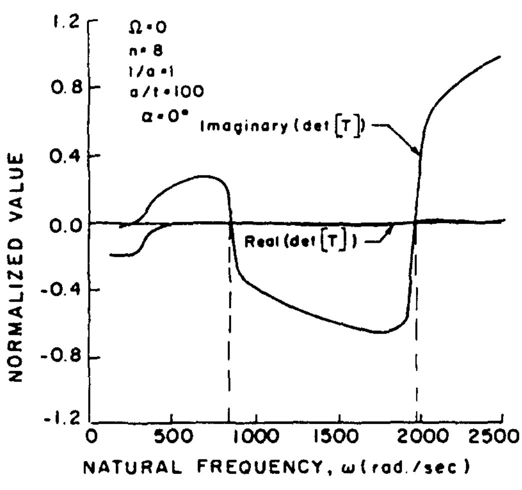


Fig. 2. Typical variation of the boundary matrix determinant with frequency.

RESULTS AND DISCUSSION

The following section contains results of a parametric investigation of spinning laminated filament-wound cylindrical shells. Ultra-high modulus graphite–epoxy has been chosen for the following examples. The material properties are:

$$E_{11} = 3\times 10^{11}\,\mathrm{N\,m^{-2}}$$
$$E_{22} = 6.2\times 10^{9}\,\mathrm{N\,m^{-2}}$$
$$E_{66} = 4.1\times 10^{9}\,\mathrm{N\,m^{-2}}$$
$$\nu_{12} = 0.26$$
$$\rho = 1.6\times 10^{3}\,\mathrm{kg\,m^{-3}}.$$

(i) *Single-layered anisotropic shells*

In the case of the non-vanishing spinning velocity, the solution has two branches which correspond to the cases of forward- and backward-traveling modes or to the cases of positive or negative spinning velocity (which may also be interpreted as positive and negative eigenvalues). In what follows, the shell's natural frequency (ω) will be presented as a function

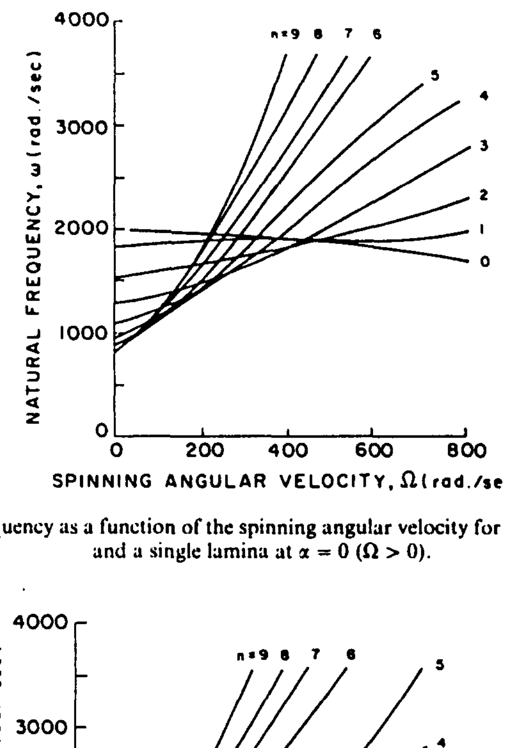


Fig. 3a. Natural frequency as a function of the spinning angular velocity for simply-supported shell and a single lamina at $\alpha = 0$ ($\Omega > 0$).

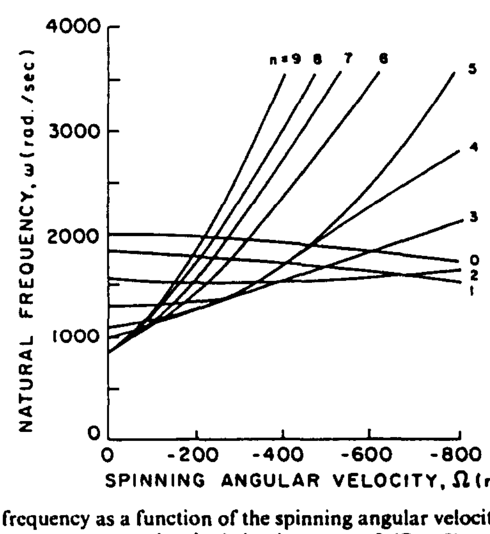


Fig. 3b. Natural frequency as a function of the spinning angular velocity for simply-supported shell and a single lamina at $\alpha = 0$ ($\Omega < 0$).

of the (positive or negative) spinning velocity (Ω), different values of circumferential mode number and laminate-winding angle (α). In all cases the shell's geometry is given by $l/a = 1$ and $t/a = 0.01$.

Figures 3a,b present the natural frequency for the case of simply-supported shells (SS3 boundary conditions, i.e. $v = w = N_\xi = M_\xi = 0$ at both ends) for a single lamina at $\alpha = 0$. Generally, by looking for the lowest natural frequency for each spinning speed, it may be seen that the higher circumferential modes dominate the behavior for low values of spinning speed. As Ω is increased, lower modes become more and more effective. As a result, the dominant modes for high values of Ω (>700 rad s^{-1}) are 0 and 1 for the cases of positive and negative spinning speeds, respectively, while the dominant modes for $\Omega = 0$ are 8 and 9 in both cases. This phenomenon results from the trend of low cirumferential modes ($n = 0$, 1) to decrease with spinning speed and the opposite trend observed in the case of high circumferential modes ($n > 3$). It should be noted that as expected for $n = 0$ the results are invariant to the sign of the spinning velocity.

Figures 4a,b present similar results obtained for these simply-supported boundary conditions for orientation angle $\alpha = 90°$. Examination of these results shows several interesting trends. First, for the case of $\Omega > 0$, as α is increased, all modes begin to show a

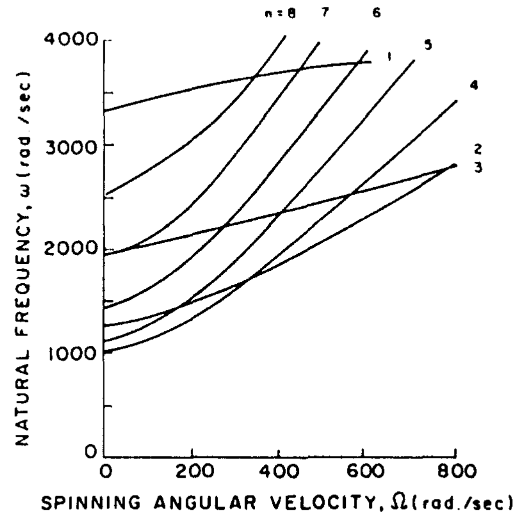


Fig. 4a. Natural frequency as a function of the spinning angular velocity for simply-supported shell and a single lamina at $\alpha = 90°$ ($\Omega > 0$).

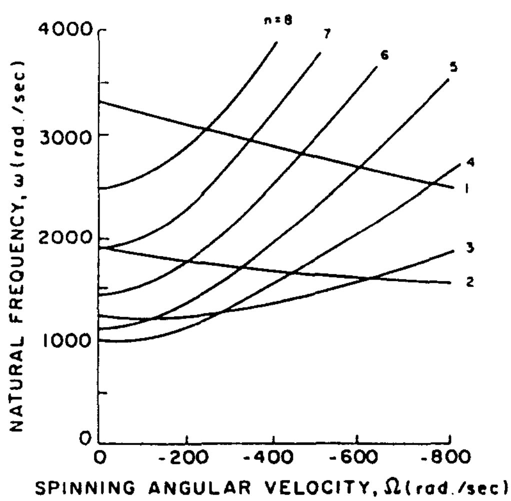


Fig. 4b. Natural frequency as a function of the spinning angular velocity for simply-supported shell and a single lamina at $\alpha = 90°$ ($\Omega < 0$).

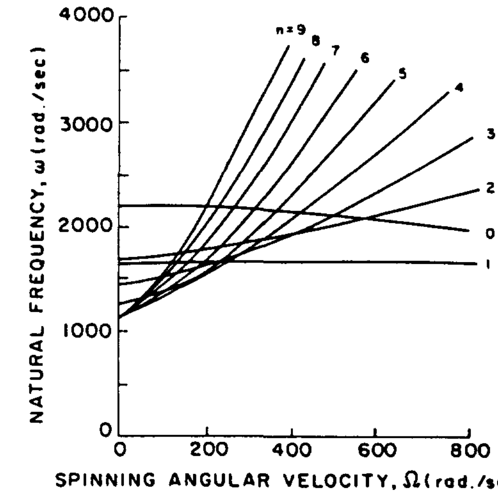


Fig. 5a. Natural frequency as a function of the spinning angular velocity for simply-supported shell and a single lamina at $\alpha = 30^\circ$ ($\Omega > 0$).

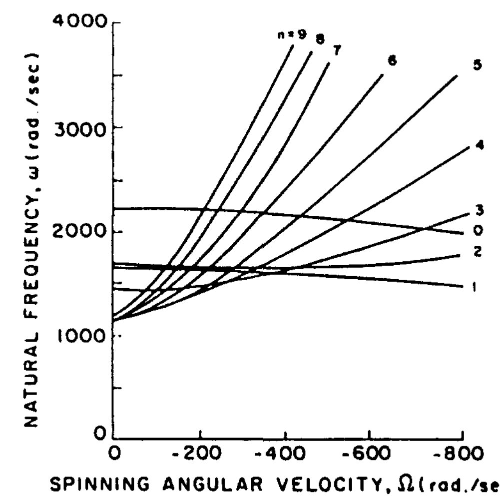


Fig. 5b. Natural frequency as a function of the spinning angular velocity for simply-supported shell and a single lamina at $\alpha = 30^\circ$ ($\Omega < 0$).

tendency to increase their natural frequency with speed. In addition, enlarging α tends to increase the natural frequency of all the modes while the lower modes are more influenced. Figures 5a,b present an intermediate case ($\alpha = 30^\circ$) where similar characteristics may be observed.

Following the replacement of the boundary conditions by the fully-clamped case RF4 (i.e. $u = v = w = w_{,x} = 0$ at both ends), additional similar calculations of the shell's natural frequency as a function of the spinning speed have been carried out. Figures 6a,b show the corresponding case of $\alpha = 0$. Compared with the "simply-supported" case (Figs 3a,b), one may observe the relatively-high natural frequency in this case. The trend of the high circumferential modes to increase with the natural frequency is also noted in this case.

(ii) *Laminated anisotropic shells*

Additional cases of shell structures which consist of three layers of different thicknesses and winding angles were examined for the case of clamped boundary conditions. In these cases the shell was assumed to be three-ply, an inner ply of thickness $h/2$ oriented at $\alpha = 0^\circ$ and two outer-plies of thicknesses $h \cdot d$ and $h(\frac{1}{2} - d)$, respectively, both oriented at $\alpha = 90^\circ$. The natural frequencies in these cases are shown in Figs 7a,b as functions of d ($0 \leqslant d \leqslant 0.5$) along with the corresponding circumferential modes for some values of d. Studying the case

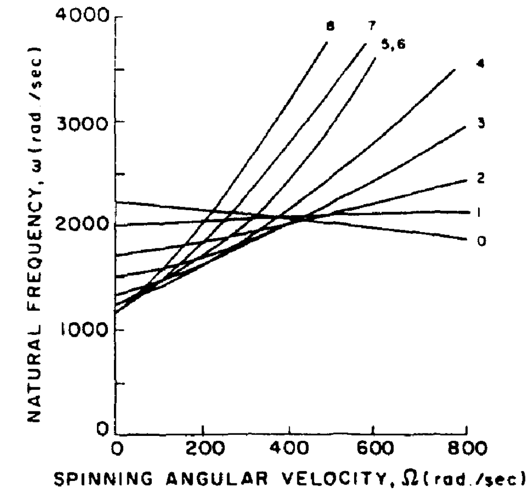


Fig. 6a. Natural frequency as a function of the spinning angular velocity for clamped shell and a single lamina at $x = 0$ ($\Omega > 0$).

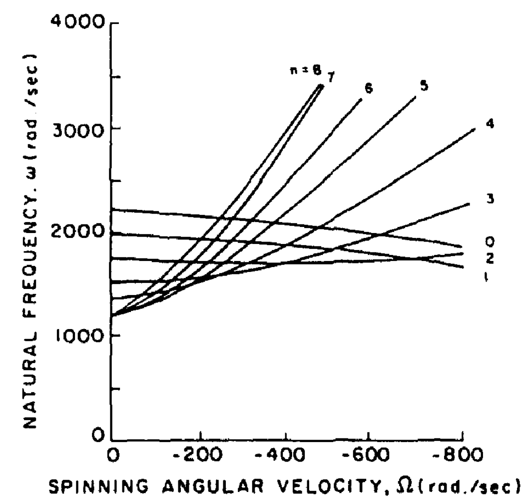


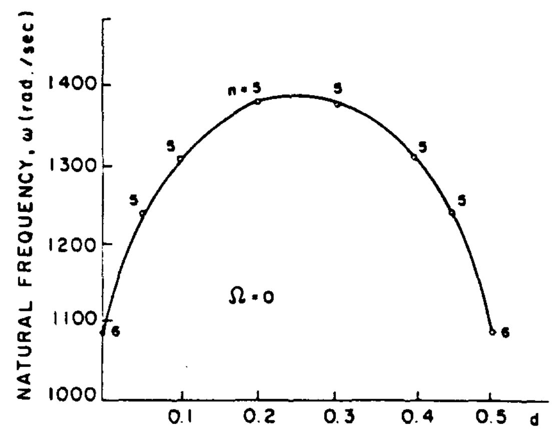
Fig. 6b. Natural frequency as a function of the spinning angular velocity for clamped shell and a single lamina at $x = 0$ ($\Omega < 0$).

of $\Omega = 0$ (Fig. 7a) shows significant changes as a function of d in the natural frequency which attains, for $d = 0.25$, a value of about 30% higher than that obtained for the two-layer shells of $d = 0$ and 0.5. However, examination of these variations in the case of $\Omega = 200$ rad s^{-1} Fig. 7b shows that the differences between the cases of $d = 0$ and $d = 0.25$ are only 7%. This is due to the fact that spinning becomes the dominant component in the determination of the shell "equivalent stiffness" and since the material density is not changed with d, the changes in natural frequencies are small.

The natural frequencies for some other simply-supported shell constructions as functions of the spinning velocity are presented in Fig. 8. Note that except for the case of a single laminate at zero fiber orientation (0, 1, 0), the characteristics of all other cases show a monotonic growth of the natural frequencies with the spinning velocity.

CONCLUDING REMARKS

The study of dynamic characteristics of rotating laminated filament-wound cylindrical shells has been presented by using a closed-form solution of a general type of field equations and arbitrary boundary conditions.



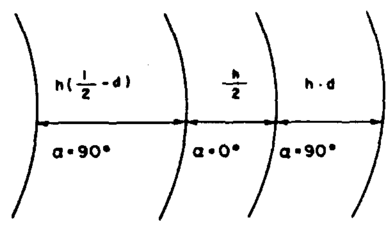


Fig. 7a. Natural frequency as a function of lamination for non-rotating clamped shell.

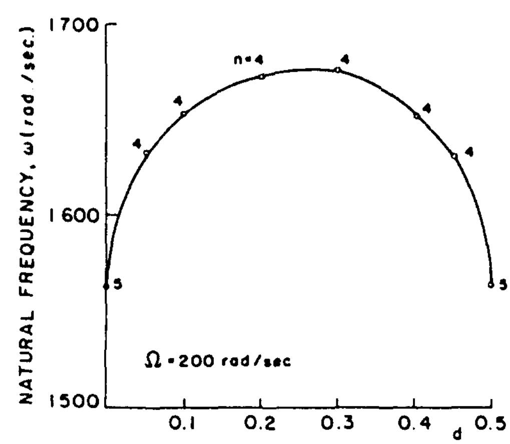


Fig. 7b. Natural frequency as a function of lamination for rotating clamped shell.

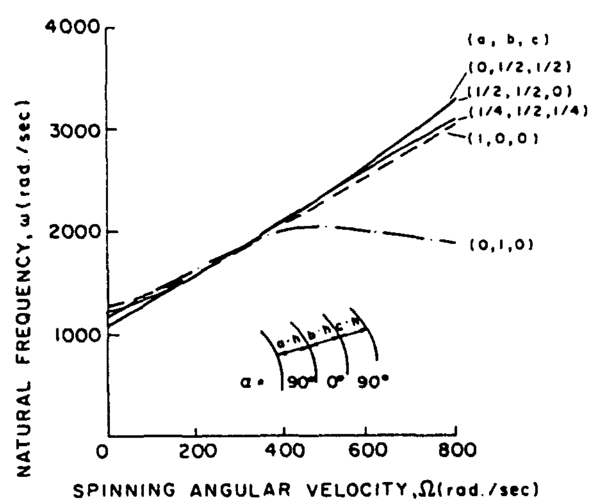


Fig. 8. Natural frequency as a function of the spinning angular velocity for simply-supported shell and different types of laminations.

The results provide an insight into the sensitivity of the resulting natural frequencies to the spinning speed and to small variations in the combinations and ordering of the laminates.

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APPENDIX A

The following are definitions of various terms that were mentioned during the derivations.

Resultant forces:

$$(N_\xi, N_\theta, N_{\theta\xi}) = \int_{h_1}^{h_2} (\tau_\xi, \tau_\theta, \tau_{\theta\xi})\, \mathrm{d}\zeta \tag{A1}$$

$$(Q_\xi, Q_\theta) = \int_{h_1}^{h_2} (\tau_{\xi\zeta}, \tau_{\theta\zeta})\, \mathrm{d}\zeta. \tag{A2}$$

Resultant moments:

$$(M_\xi, M_\theta, M_{\theta\xi}) = \int_{h_1}^{h_2} (\tau_\xi, \tau_\theta, \tau_{\theta\xi})\zeta\, \mathrm{d}\zeta. \tag{A3}$$

Inertia constants:

$$(R_0, R_1, R_2) = \int_{h_1}^{h_2} (1, \zeta, \zeta^2)\rho\, \mathrm{d}\zeta. \tag{A4}$$

Rotational accelerations:

$$a_\xi, a_\theta, a_\zeta = \frac{1}{R_0}\int_{h_1}^{h_2} (a_\xi, a_\theta, -a_r)\, \rho\, \mathrm{d}\zeta. \tag{A5}$$

The elements of the matrix in eqn (8):

$$(A_{ij}, B_{ij}, D_{ij}) = \int (1, \zeta, \zeta^2) E_{ij}\, \mathrm{d}\zeta \quad (i,j = \xi, \theta, \zeta). \tag{A6}$$

APPENDIX B

The operators L_{Sij} are given by:

$$L_{S11} = -n^2 A_{\theta\theta}(\ ) + 2inA_{\theta\xi}(\ )_{,\xi} + A_{\xi\xi}(\ )_{,\xi\xi} \tag{B1}$$

$$L_{S12} = -n^2\left(A_{\theta\theta} - \frac{1}{a}B_{\theta\theta}\right)(\ ) + in\left[A_{\xi\xi} + A_{\xi\theta} - \frac{1}{a}(B_{\xi\theta} + 2B_{\theta\xi})\right](\ )_{,\xi} + \left(A_{\xi\theta} - \frac{2}{a}B_{\xi\theta}\right)(\ )_{,\xi\xi} \tag{B2}$$

$$L_{S13} = in\left(n^2\frac{1}{a}B_{\theta\theta} - A_{\theta\theta}\right)(\ ) + \left[n^2\frac{1}{a}(2B_{\theta\xi} + B_{\xi\theta}) - A_{\xi\theta}\right](\ )_{,\xi} - 3in\frac{1}{a}B_{\theta\xi}(\ )_{,\xi\xi} - \frac{1}{a}B_{\xi\xi}(\ )_{,\xi\xi\xi} \tag{B3}$$

$$L_{S21} = -n^2\left(A_{\theta\theta} - \frac{1}{a}B_{\theta\theta}\right)(\ ) + in\left[A_{\theta\xi} + A_{\xi\theta} - \frac{1}{a}(B_{\xi\theta} + B_{\theta\xi})\right](\ )_{,\xi} + \left(A_{\theta\xi} - \frac{1}{a}B_{\theta\xi}\right)(\ )_{,\xi\xi} \tag{B4}$$

$$L_{S22} = -n^2\left(A_{\theta\theta} - \frac{2}{a}B_{\theta\theta} + \frac{1}{a^2}D_{\theta\theta}\right)(\ ) + in\left(2A_{\theta\xi} - \frac{5}{a}B_{\theta\xi} + \frac{3}{a^2}D_{\theta\xi}\right)(\ )_{,\xi} + \left(A_{\xi\xi} - \frac{3}{a}B_{\xi\xi} + \frac{2}{a^2}D_{\xi\xi}\right)(\ )_{,\xi\xi} \tag{B5}$$

$$L_{S23} = in\left[-n^2\frac{1}{a}\left(-B_{\theta\theta} + \frac{1}{a}D_{\theta\theta}\right) - A_{\theta\theta} + \frac{1}{a}B_{\theta\theta}\right](\ )\left[3n^2\frac{1}{a}\left(B_{\theta\xi} - \frac{1}{a}D_{\theta\xi}\right) - A_{\xi\theta} + \frac{1}{a}B_{\xi\theta}\right](\ )_{,\xi}$$
$$- in\frac{1}{a}\left[B_{\theta\xi} + 2B_{\xi\xi} - \frac{1}{a}(D_{\theta\xi} + 2D_{\xi\xi})\right](\ )_{,\xi\xi} + \frac{1}{a}\left(-B_{\xi\xi} + \frac{1}{a}D_{\xi\xi}\right)(\ )_{,\xi\xi\xi} \tag{B6}$$

$$L_{S31} = L_{S13} \tag{B7}$$

$$L_{S32} = in\left[-n^2\frac{1}{a}\left(-B_{\theta\theta} + \frac{1}{a}D_{\theta\theta}\right) - A_{\theta\theta} + \frac{1}{a}B_{\theta\theta}\right](\ )\left[n^2\frac{1}{a}\left(3B_{\xi\theta} - \frac{4}{a}D_{\xi\theta}\right) - A_{\theta\theta} + \frac{2}{a}B_{\theta\theta}\right](\ )_{,\xi}$$
$$- in\frac{1}{a}\left[2B_{\theta\theta} + B_{\xi\theta} - \frac{1}{a}(4D_{\xi\xi} + D_{\xi\theta})\right](\ )_{,\xi\xi} - \frac{1}{a}\left(B_{\xi\theta} - \frac{2}{a}D_{\xi\xi}\right)(\ )_{,\xi\xi\xi} \tag{B8}$$

$$L_{S33} = \left[\frac{n^4}{a^2}D_{\theta\theta} - 2n^2\frac{1}{a}B_{\theta\theta} + A_{\theta\theta}\right](\ ) + \left(\frac{4}{a^2}D_{\theta\xi} + in\frac{4}{a}B_{\theta\xi}\right)(\ )_{,\xi} + \left[-\frac{2n^2}{a^2}(2D_{\xi\xi} + D_{\theta\xi}) - \frac{2}{a}B_{\theta\xi}\right](\ )_{,\xi\xi}$$
$$+ in\frac{4}{a^2}D_{\theta\xi}(\ )_{,\xi\xi\xi} + \frac{1}{a^2}D_{\xi\xi}(\ )_{,\xi\xi\xi\xi}. \tag{B9}$$

The operators L_{Dij} are:

$$L_{D11} = a^2 R_0 \omega^2(\ ) \tag{B10}$$

$$L_{D12} = -2iaR_1\omega\Omega(\ )_{,\xi} - a^2R_0\Omega^2 in(\ )_{,\xi} \tag{B11}$$

$$L_{D13} = -aR_1(\omega^2 + \Omega^2)(\ )_{,\xi} + a^2R_0\Omega^2(\ )_{,\xi} \tag{B12}$$

$$L_{D21} = R_0 a^2\Omega^2 in(\ )_{,\xi} \tag{B13}$$

$$L_{D22} = (a^2R_0 - 3aR_1 + 2R_2)(\omega^2 + \Omega^2)(\ ) - (R_2 - aR_1)\,2\omega n\Omega(\ ) \tag{B14}$$

$$L_{D23} = (a^2R_0 - 3aR_1 + 2R_2)2i\omega\Omega(\ ) - in(R_2 - aR_1)(\omega^2 + \Omega^2)(\ ) \tag{B15}$$

$$L_{D31} = -R_1 a\omega^2(\ )_{,\xi} \tag{B16}$$

$$L_{D32} = 2inR_2(\omega^2 + \Omega^2)(\ ) + R_2n^2 + R_0a^2)2i\omega\Omega(\ ) - R_0a^2\Omega^2 in(\ ) - R_2 2i\omega\Omega(\ )_{,\xi\xi} \tag{B17}$$

$$L_{D33} = 4n\omega R_2\Omega(\ ) - (R_2n^2 + R_0a^2)(\omega^2 + \Omega^2)(\ ) + R_0a^2\Omega^2n^2(\ ) + R_2(\omega^2 + \Omega^2)(\ )_{,\xi\xi}. \tag{B18}$$